Flexible sensitivity analysis for causal inference in observational studies subject to unmeasured confounding

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Overview

Background: causal inference in observational studies

Sensitivity analysis with unmeasured confounding

Extensions

Potential outcome framework

- Potential outcome framework.
- Potential outcomes: $Y_i(1)$ and $Y_i(0)$.
- Binary treatment: Z_i.
- Observed outcome: $Y_i = Y_i(Z_i) = Z_i Y_i(1) + (1 - Z_i) Y_i(0)$.
- Stable unit treatment values assumption.
- Super population regime: independently and identically distributed $\{X_i, Z_i, Y_i(1), Y_i(0) : i = 1, \dots, n\}$.

Parameter of interest

- Causal parameter of interest: $\tau = E\{Y(1) - Y(0)\}$, the average treatment effect, decomposes into

$$\begin{aligned} \tau &= [E(Y \mid Z=1)\mathrm{pr}(Z=1) + E\{Y(1) \mid Z=0\}\mathrm{pr}(Z=0)] \\ &\quad - [E\{Y(0) \mid Z=1\}\mathrm{pr}(Z=1) + E(Y \mid Z=0)\mathrm{pr}(Z=0)]\,. \end{aligned}$$

- Fundamental challenge of causal inference: to estimate the counterfactual means $E\{Y(1) \mid Z=0\}$ and $E\{Y(0) \mid Z=1\}$.
- Randomization leads to obvious identification, but in observational studies?

Identification under unconfoundedness

- Unconfoundedness assumption: $Z \perp\!\!\!\perp \{Y(1), Y(0)\} \mid X$ (Rosenbaum and Rubin, 1983).
- Under this assumption, $E\{Y(1) \mid Z=1, X\} = E\{Y(1) \mid Z=0, X\}$, thus τ is nonparametrically identified.
- Two identification formulas:

$$\begin{aligned}\tau &= E\{\mu_1(X) - \mu_0(X)\} \\ &= E\left\{\frac{ZY}{e(X)} - \frac{(1-Z)Y}{1-e(X)}\right\},\end{aligned}$$

  where
  - $\mu_1(X) = E(Y \mid Z=1, X)$ and $\mu_0(X) = E(Y \mid Z=0, X)$: conditional expectation of outcomes;
  - $e(X) = \mathrm{pr}(Z=1 \mid X)$: propensity score.
- Implicitly assume overlap: $0 < e(X) < 1$.

Estimation under unconfoundedness

- Estimators corresponding to the two identification formulas:

$$\hat{\tau}^{\text{reg}} = n^{-1} \sum_{i=1}^{n} \{\hat{\mu}_1(X_i) - \hat{\mu}_0(X_i)\},$$

$$\hat{\tau}^{\text{ht}} = n^{-1} \sum_{i=1}^{n} \left\{ \frac{Z_i Y_i}{\hat{e}(X_i)} - \frac{(1 - Z_i) Y_i}{1 - \hat{e}(X_i)} \right\},$$

where $\hat{e}(X_i)$ and $\hat{\mu}_z(X_i)$ are fitted propensity score and outcome models.

Estimation under unconfoundedness

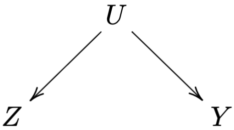
- Doubly robust estimator by combining both models (Bang and Robins, 2005):

$$\hat{\tau}^{\mathrm{dr}} = \hat{\tau}^{\mathrm{reg}} + n^{-1} \sum_{i=1}^{n} \left[\frac{Z_i \{Y_i - \hat{\mu}_1(X_i)\}}{\hat{e}(X_i)} - \frac{(1 - Z_i)\{Y_i - \hat{\mu}_0(X_i)\}}{1 - \hat{e}(X_i)} \right].$$

- Modifies $\hat{\tau}^{\mathrm{reg}}$ by inverse propensity score weighted residuals.
- Consistent to τ if either outcome models or propensity score model is correctly specified.

Sensitivity analysis

- Unconfoundedness assumption: untestable, cannot use data to validate.
- Existence of unmeasured confounding possibly overturns an observed association between the treatment and outcome.
- Hidden confounder U:



Source: Ding (2024).

- Sensitivity analysis: assess the impact of U; how strong the unmeasured confounding needs to be to overturn the observed association.

Sensitivity analysis

- Parametric models to assess the impact of U on the estimation of τ (Rosenbaum and Rubin, 1983; Lin et al., 1998; Imbens, 2003).
- Sensitivity analysis to test the sharp null hypothesis of no unit-level causal effects in matched-pair observational studies (Rosenbaum, 1987).
- E-value: sensitivity analysis for causal estimates based on risk ratios (Cornfield et al., 1959; Ding and VanderWeele, 2016; VanderWeele and Ding, 2017).
- Sensitivity analysis methods for the inverse propensity score weighting estimator (Zhao et al., 2019; Dorn and Guo, 2022).
- Useful for specific estimation or testing strategies.
- Deal with the standard estimators $\hat{\tau}^{\text{reg}}$, $\hat{\tau}^{\text{ht}}$ and $\hat{\tau}^{\text{dr}}$ simultaneously?

Identification challenge revisit: two extreme solutions

$$\begin{aligned}\tau &= [E(Y \mid Z=1)\mathrm{pr}(Z=1) + E\{Y(1) \mid Z=0\}\mathrm{pr}(Z=0)] \\ &\quad - [E\{Y(0) \mid Z=1\}\mathrm{pr}(Z=1) + E(Y \mid Z=0)\mathrm{pr}(Z=0)]\,.\end{aligned}$$

- Unconfoundedness assumption: $Z \perp\!\!\!\perp \{Y(1), Y(0)\} \mid X$.

$$\begin{aligned}E\{Y(1) \mid Z=1, X\} &= E\{Y(1) \mid Z=0, X\}, \\ E\{Y(0) \mid Z=1, X\} &= E\{Y(0) \mid Z=0, X\}.\end{aligned}$$

- Very restrictive—assumes the two treatment groups have **identical** conditional means.

Identification challenge revisit: two extreme solutions

$$
\begin{aligned}
\tau = & \; [E(Y \mid Z=1)\text{pr}(Z=1) + E\{Y(1) \mid Z=0\}\text{pr}(Z=0)] \\
& - [E\{Y(0) \mid Z=1\}\text{pr}(Z=1) + E(Y \mid Z=0)\text{pr}(Z=0)] \,.
\end{aligned}
$$

- Partial identification method:
  - Assume the potential outcomes are bounded between $[\ell, u]$.
  - $E\{Y(1)\}$ has lower bound

    $$E(Y \mid Z=1)\text{pr}(Z=1) + \ell\text{pr}(Z=0)$$

    and upper bound

    $$E(Y \mid Z=1)\text{pr}(Z=1) + u\text{pr}(Z=0).$$

  - Similar lower and upper bounds for $E\{Y(0)\}$, leading to bounds for τ.
  - Problem: bounds are always too wide to provide valuable causal information.
- Is there a midground?

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Sensitivity analysis with unmeasured confounding

- Define sensitivity parameters:

$$\frac{E\{Y(1) \mid Z=1,X\}}{E\{Y(1) \mid Z=0,X\}} = \varepsilon_1(X), \quad \frac{E\{Y(0) \mid Z=1,X\}}{E\{Y(0) \mid Z=0,X\}} = \varepsilon_0(X).$$

- $\varepsilon_1(X)$ and $\varepsilon_0(X)$: two sensitivity parameters.
- Quantifies the violation of the unconfoundedness assumption.
- $\varepsilon_1(X) = \varepsilon_0(X) = 1$: the unconfoundedness assumption.
- First fix them to obtain the corresponding estimators and then vary them within a range to obtain a sequence of estimators.

Identification and estimation under sensitivity analysis

Outcome regression

- With known $\varepsilon_1(X)$ and $\varepsilon_0(X)$,

$$
\begin{aligned}
E\{Y(1) \mid Z=0\} &= E\{\mu_1(X)/\varepsilon_1(X) \mid Z=0\}, \\
E\{Y(0) \mid Z=1\} &= E\{\mu_0(X)\varepsilon_0(X) \mid Z=1\}.
\end{aligned}
$$

- Estimator:

$$
\begin{aligned}
\hat{\tau}^{\text{proj}} &= n^{-1}\sum_{i=1}^{n}\{Z_i\hat{\mu}_1(X_i) + (1-Z_i)\hat{\mu}_1(X_i)/\varepsilon_1(X_i)\} \\
&\quad - n^{-1}\sum_{i=1}^{n}\{Z_i\hat{\mu}_0(X_i)\varepsilon_0(X_i) + (1-Z_i)\hat{\mu}_0(X_i)\}.
\end{aligned}
$$

Identification and estimation under sensitivity analysis

Inverse propensity score weighting

- With known $\varepsilon_1(X)$ and $\varepsilon_0(X)$,

$$
\begin{aligned}
E\{Y(1)\} &= E\left\{w_1(X)\frac{Z}{e(X)}Y\right\}, \\
E\{Y(0)\} &= E\left\{w_0(X)\frac{1-Z}{1-e(X)}Y\right\},
\end{aligned}
$$

  where

$$
w_1(X) = e(X)+\{1-e(X)\}/\varepsilon_1(X),\ w_0(X) = e(X)\varepsilon_0(X)+1-e(X).
$$

- Estimator:

$$
\hat{\tau}^{\mathrm{ht}} = n^{-1}\sum_{i=1}^{n}\hat{w}_1(X_i)\frac{Z_iY_i}{\hat{e}(X_i)} - n^{-1}\sum_{i=1}^{n}\hat{w}_0(X_i)\frac{(1-Z_i)Y_i}{1-\hat{e}(X_i)}.
$$

Identification and estimation under sensitivity analysis

- When $\varepsilon_1(X) = \varepsilon_0(X) = 1$, reduces to the unconfoundedness assumption. Recover previous results.
- Motivates a combined strategy: doubly robust estimation (Bang and Robins, 2005; Bickel et al., 1993).

Doubly robust

- Outcome regression

$$
\begin{aligned}
\hat{\tau}^{\text{proj}} &= n^{-1}\sum_{i=1}^{n}\{Z_i\hat{\mu}_1(X_i) + (1 - Z_i)\hat{\mu}_1(X_i)/\varepsilon_1(X_i)\} \\
&\quad - n^{-1}\sum_{i=1}^{n}\{Z_i\hat{\mu}_0(X_i)\varepsilon_0(X_i) + (1 - Z_i)\hat{\mu}_0(X_i)\}.
\end{aligned}
$$

- IPW

$$
\hat{\tau}^{\text{ht}} = n^{-1}\sum_{i=1}^{n}\hat{w}_1(X_i)\frac{Z_iY_i}{\hat{e}(X_i)} - n^{-1}\sum_{i=1}^{n}\hat{w}_0(X_i)\frac{(1 - Z_i)Y_i}{1 - \hat{e}(X_i)}.
$$

Efficient influence function

- Under our definition of sensitivity parameters, the efficient influence functions for $E\{Y(1)\}$ and $E\{Y(0)\}$ are respectively

$$\begin{aligned}\phi_1 &= w_1(X)\frac{Z}{e(X)}Y - \frac{\{Z-e(X)\}\mu_1(X)}{e(X)\varepsilon_1(X)} - E\{Y(1)\},\\ \phi_0 &= w_0(X)\frac{1-Z}{1-e(X)}Y - \frac{\{e(X)-Z\}\mu_0(X)\varepsilon_0(X)}{1-e(X)} - E\{Y(0)\},\end{aligned}$$

  so the efficient influence function for τ is $\phi_1 - \phi_0$.
- An estimator constructed based on the EIF:

$$\begin{aligned}\hat{\tau}^{\mathrm{dr}} &= n^{-1}\sum_{i=1}^{n}\left[\hat{w}_1(X_i)\frac{Z_iY_i}{\hat{e}(X_i)} - \frac{\{Z_i-\hat{e}(X_i)\}\,\hat{\mu}_1(X_i)}{\hat{e}(X_i)\varepsilon_1(X_i)}\right]\\ &\quad -n^{-1}\sum_{i=1}^{n}\left[\hat{w}_0(X_i)\frac{(1-Z_i)Y_i}{1-\hat{e}(X_i)} - \frac{\{\hat{e}(X_i)-Z_i\}\,\hat{\mu}_0(X_i)\varepsilon_0(X_i)}{1-\hat{e}(X_i)}\right].\end{aligned}$$

- Can be written as modifications of $\hat{\tau}^{\mathrm{proj}}$ and $\hat{\tau}^{\mathrm{ht}}$.

Double robustness and semiparametric efficiency

$$
\begin{aligned}
\hat{\tau}^{\mathrm{dr}} \;=\; & n^{-1}\sum_{i=1}^{n}\left[\hat{w}_1(X_i)\frac{Z_iY_i}{\hat{e}(X_i)} - \frac{\{Z_i-\hat{e}(X_i)\}\,\hat{\mu}_1(X_i)}{\hat{e}(X_i)\varepsilon_1(X_i)}\right] \\
& -n^{-1}\sum_{i=1}^{n}\left[\hat{w}_0(X_i)\frac{(1-Z_i)Y_i}{1-\hat{e}(X_i)} - \frac{\{\hat{e}(X_i)-Z_i\}\,\hat{\mu}_0(X_i)\varepsilon_0(X_i)}{1-\hat{e}(X_i)}\right].
\end{aligned}
$$

- Double robustness: consistent if either the propensity score or the outcome model is correctly specified.
- Semiparametric efficiency bound is achieved with
  1. consistency of both models,
  2. mild requirements on their convergence rates.

Implementation—calibration of sensitivity parameters

- Observed data do not provide information on sensitivity parameters.
- How can we make meaningful progress?
  - A standard strategy: leave one covariate out.
  - Pretending an observed covariate is an unmeasured confounder.
  - Assume ignorability $Z \perp\!\!\!\perp \{Y(1), Y(0)\} \mid X$ and calculate

$$\varepsilon_z(X_{-j}) = \frac{E\{Y(z) \mid Z = 1, X_{-j}\}}{E\{Y(z) \mid Z = 0, X_{-j}\}}, \quad (z = 0, 1).$$

  - Use the range of $\hat{\varepsilon}_z(X_{-j})$ to specify the range of $(\varepsilon_1(X), \varepsilon_0(X))$.

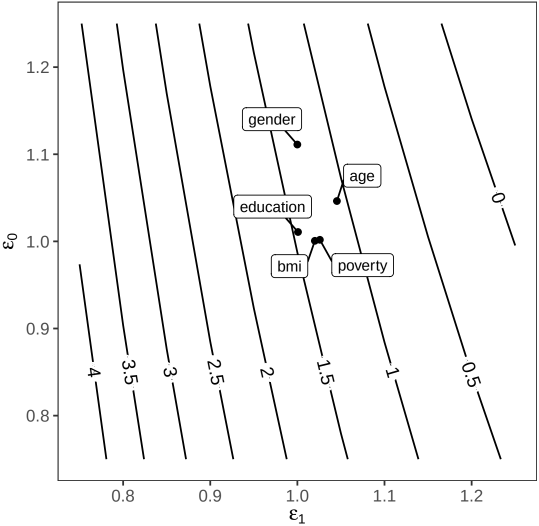
Sensitivity parameters: vary them with covariates or not

- Our theory allows for the dependence of sensitivity parameters on X.
  - Complicated in implementation.
  - Not easy to visualize the sensitivity analysis results.
- Practically, specify $(\varepsilon_1(X), \varepsilon_0(X)) = (\varepsilon_1, \varepsilon_0)$ independent of X.
  - Point estimates are monotonic in $\varepsilon_1, \varepsilon_0$: easy to interpret.
  - For non-negative outcomes: can also be interpreted as the worst-case result even with $(\varepsilon_1(X), \varepsilon_0(X))$ depending on X.
  - Formal result in the paper.

Demo with an example

- Re-analyze the observational study in Bazzano et al. (2003): whether cigarette smoking has a causal effect on homocysteine levels.
- Elevation of homocysteine level is a risk factor for cardiovascular disease.
- Data: the U.S. National Health and Nutrition Examination Survey 2005– 2006.
- Observed covariates: gender, age, education level, body mass index (BMI), and poverty.
- $\hat{\tau}^{\mathrm{dr}}$ for τ: 1.48 with a 95% confidence interval (0.78, 2.18).
- Unobserved confounders: genotype?

Demo with an example – visualization based on $\hat{\tau}^{\mathrm{dr}}$



- Point estimates as a function of $(\varepsilon_1, \varepsilon_0)$.
- Plot maximums of $\hat{\varepsilon}_z(X_{-j})$ for observed covariates.

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Extension to nonlinear causal parameters

- A more general class of nonlinear causal parameters $g(\mu_1, \mu_0)$.
- Previous results: $g(\mu_1, \mu_0) = \mu_1 - \mu_0$.
- Binary outcomes: the causal risk ratio and the causal odds ratio,
$$\mathrm{RR} = \frac{\mu_1}{\mu_0} \quad \text{and} \quad \mathrm{OR} = \frac{\mu_1/(1-\mu_1)}{\mu_0/(1-\mu_0)}.$$
- Plug-in estimators $g(\hat{\mu}_1^*, \hat{\mu}_0^*)$ for $* \in \{\text{pred, prod, ht, dr}\}$.

Extension to bias-corrected matching estimator

- Matching: impute the missing counterfactual $Y_i(0)$ for $Z_i = 1$ by finding M nearest neighbors of i in the control group.
- Use the average observed outcomes of the M nearest neighbors as the imputed value of the counterfactual $Y_i(0)$. Similar procedure for $Y_i(1)$.
- Matching-based estimator: average of the imputed individual treatment effect. Generally inconsistent (Abadie and Imbens, 2006).
- Abadie and Imbens (2011) proposed a bias-corrected version of the matching estimator by estimating the conditional outcome models $\mu_z(X)$ and combining it with the original matching estimator.

Extension to bias-corrected matching estimator

- Rewrite the bias-corrected matching estimator as (Lin et al., 2023):

$$\begin{aligned}\hat{\tau}_M^{\mathrm{bc}} &= \hat{\tau}^{\mathrm{reg}} + n^{-1}\sum_{i=1}^{n}\left\{1+\frac{K_M^1(X_i)}{M}\right\} Z_i \left\{Y_i - \hat{\mu}_1(X_i)\right\} \\ &\quad - n^{-1}\sum_{i=1}^{n}\left\{1+\frac{K_M^0(X_i)}{M}\right\}(1-Z_i)\left\{Y_i - \hat{\mu}_0(X_i)\right\},\end{aligned}$$

  where M is the fixed number of matches for each observation and $K_M^z(X_i)$ is the number of matched times of unit i in treatment group z, $z \in \{0, 1\}$.

Extension to bias-corrected matching estimator

- Under our sensitivity analysis framework, the bias-corrected matching estimator:

$$\begin{aligned}\hat{\tau}_M^{\mathrm{bc}} &= \hat{\tau}^{\mathrm{pred}} + n^{-1}\sum_{i=1}^{n}\frac{K_M^1(X_i)}{M}\frac{1}{\varepsilon_1(X_i)}Z_i\left\{Y_i - \hat{\mu}_1(X_i)\right\} \\ &\quad - n^{-1}\sum_{i=1}^{n}\frac{K_M^0(X_i)}{M}\varepsilon_0(X_i)(1-Z_i)\left\{Y_i - \hat{\mu}_0(X_i)\right\}.\end{aligned}$$

- Lin et al. (2023) views matching as a nonparametric method of estimating the propensity score. We essentially use $1 + K_M^1(X_i)/M$ and $1 + K_M^0(X_i)/M$ to estimate $1/e(X_i)$ and $1/\{1 - e(X_i)\}$, respectively.

Extension to multi-level treatment

- Observational studies with a multi-level treatment, $Z \in \{1, \ldots, K\}$.
- Each unit has K potential outcomes $\{Y(1), \ldots, Y(K)\}$ corresponding to the K treatment levels.
- Causal parameters of interest: comparisons of potential outcomes
$$\tau_c = \sum_{k=1}^{K} c_k E\{Y(k)\},$$
where $\sum_{k=1}^{K} c_k = 0$.
- For any two treatment levels k and l, the sensitivity parameters:
$$\varepsilon_{k,l}(X) = \frac{E\{Y(k) \mid Z = k, X\}}{E\{Y(k) \mid Z = l, X\}}.$$

Extension to multi-level treatment

- Identification:

$$\begin{aligned} E\{Y(k)\} &= \sum_{l=1}^{K} E\left\{1(Z=l)\frac{\mu_k(X)}{\varepsilon_{k,l}(X)}\right\} \\ &= \sum_{l=1}^{K} E\left\{\frac{e_l(X)}{\varepsilon_{k,l}(X)}\frac{1(Z=k)Y}{e_k(X)}\right\}, \end{aligned}$$

  where
  - $e_k(X) = \mathrm{pr}(Z = k \mid X)$: the generalized propensity score (Imbens, 2000; Imai and Van Dyk, 2004);
  - $\mu_k(X) = E\{Y \mid Z = k, X\}$: conditional outcome mean.
- Estimation:

$$\begin{aligned} \hat{\mu}_k^{\mathrm{dr}} &= \hat{\mu}_k^{\mathrm{reg}} + n^{-1}\sum_{i=1}^{n}\sum_{l=1}^{K}\frac{\hat{e}_l(X_i)1(Z_i=k)\{Y_i-\hat{\mu}_k(X_i)\}}{\varepsilon_{k,l}(X_i)\hat{e}_k(X_i)}, \\ \hat{\tau}_c^{dr} &= \sum_{k=1}^{K} c_k \hat{\mu}_k^{dr}. \end{aligned}$$

Other extensions in the paper

- Hajek-type weighting estimators.
- Average treatment effect on the treated:

$$\tau_{\mathrm{T}} = E\{Y(1) - Y(0) \mid Z = 1\} = E(Y \mid Z = 1) - E\{Y(0) \mid Z = 1\}.$$

- Survival outcomes under right-censoring:

$$\tau(t) = S_1(t) - S_0(t),$$

where $S_z(t) = \mathrm{pr}\{Y(z) > t\}$ denotes the potential survival functions for $z \in \{0, 1\}$.

- Controlled direct effect.
- Discussion on another sensitivity parameter, difference scale.

Summary

- **Flexible** sensitivity analysis framework.
- Simultaneously deal with weighting, outcome regression, and doubly robust estimators.
- Only requires simple modifications of the standard estimators.
- Extends to many other causal inference settings.
- Easy to implement – R package `saci`.

Thank you very much!

ArXiv: `https://arxiv.org/abs/2305.17643`

Abadie, A. and Imbens, G. W. (2006). Large sample properties of matching estimators for average treatment effects. *Econometrica* **74,** 235–267.

Abadie, A. and Imbens, G. W. (2011). Bias-corrected matching estimators for average treatment effects. *Journal of Business and Economic Statistics* **29,** 1–11.

Bang, H. and Robins, J. M. (2005). Doubly robust estimation in missing data and causal inference models. *Biometrics* **61,** 962–973.

Bazzano, L. A., He, J., Muntner, P., Vupputuri, S., and Whelton, P. K. (2003). Relationship between cigarette smoking and novel risk factors for cardiovascular disease in the united states. *Annals of Internal Medicine* **138,** 891–897.

Bickel, P. J., Klaassen, C. A., Ritov, Y., and Wellner, J. A. (1993). *Efficient and Adaptive Estimation for Semiparametric Models*. Baltimore: Johns Hopkins University Press.

Cornfield, J., Haenszel, W., Hammond, E. C., Lilienfeld, A. M., Shimkin, M. B., and Wynder, E. L. (1959). Smoking and lung cancer: recent evidence and a discussion of some questions. *Journal of the National Cancer Institute* **22,** 173–203.

Ding, P. (2024). *A First Course in Causal Inference*. New York: Chapman & Hall.

Ding, P. and VanderWeele, T. J. (2016). Sensitivity analysis without assumptions. *Epidemiology* **27,** 368–377.

Dorn, J. and Guo, K. (2022). Sharp sensitivity analysis for inverse propensity weighting via quantile balancing. *Journal of the American Statistical Association* **00,** 1–13.

Imai, K. and Van Dyk, D. A. (2004). Causal inference with general treatment regimes: Generalizing the propensity score. *Journal of the American Statistical Association* **99,** 854–866.

Imbens, G. W. (2000). The role of the propensity score in estimating dose-response functions. *Biometrika* **87,** 706–710.

Imbens, G. W. (2003). Sensitivity to exogeneity assumptions in program evaluation. *American Economic Review* **93,** 126–132.

Lin, D. Y., Psaty, B. M., and Kronmal, R. A. (1998). Assessing the sensitivity of regression results to unmeasured confounders in observational studies. *Biometrics* **54,** 948–963.

Lin, Z., Ding, P., and Han, F. (2023). Estimation based on nearest neighbor matching: from density ratio to average treatment effect. *Econometrica* **91,** 2187–2217.

Rosenbaum, P. R. (1987). Sensitivity analysis for certain permutation inferences in matched observational studies. *Biometrika* **74,** 13–26.

Rosenbaum, P. R. and Rubin, D. B. (1983). The central role of the propensity score in observational studies for causal effects. *Biometrika* **70,** 41–55.

VanderWeele, T. J. and Ding, P. (2017). Sensitivity analysis in observational research: introducing the e-value. *Annals of Internal Medicine* **167,** 268–274.

Zhao, Q., Small, D. S., and Bhattacharya, B. B. (2019). Sensitivity analysis for inverse probability weighting estimators via the percentile bootstrap. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* **81,** 735–761.

More equivalent forms for $\hat{\tau}$ with outcome regression

$$
\begin{aligned}
\hat{\tau}^{\text{pred}} &= n^{-1}\sum_{i=1}^{n}\{Z_iY_i + (1-Z_i)\hat{\mu}_1(X_i)/\varepsilon_1(X_i)\} \\
&\quad - n^{-1}\sum_{i=1}^{n}\{Z_i\hat{\mu}_0(X_i)\varepsilon_0(X_i) + (1-Z_i)Y_i\},
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\tau}^{\text{proj}} &= n^{-1}\sum_{i=1}^{n}\{Z_i\hat{\mu}_1(X_i) + (1-Z_i)\hat{\mu}_1(X_i)/\varepsilon_1(X_i)\} \\
&\quad - n^{-1}\sum_{i=1}^{n}\{Z_i\hat{\mu}_0(X_i)\varepsilon_0(X_i) + (1-Z_i)\hat{\mu}_0(X_i)\}.
\end{aligned}
$$

More equivalent forms for $\hat{\tau}^{\mathrm{dr}}$

$$
\begin{aligned}
\hat{\tau}^{\mathrm{dr}} &= \hat{\tau}^{\mathrm{ht}} - n^{-1} \sum_{i=1}^{n} \check{Z}_i \left\{ \frac{\hat{\mu}_1(X_i)}{\hat{e}(X_i)\varepsilon_1(X_i)} + \frac{\hat{\mu}_0(X_i)\varepsilon_0(X_i)}{1-\hat{e}(X_i)} \right\} \\
&= \hat{\tau}^{\mathrm{pred}} + n^{-1} \sum_{i=1}^{n} \left\{ \frac{1-\hat{e}(X_i)}{\varepsilon_1(X_i)} \frac{Z_i \check{Y}_i}{\hat{e}(X_i)} - \hat{e}(X_i)\varepsilon_0(X_i) \frac{(1-Z_i)\check{Y}_i}{1-\hat{e}(X_i)} \right\} \\
&= \hat{\tau}^{\mathrm{proj}} + n^{-1} \sum_{i=1}^{n} \left\{ \hat{w}_1(X_i) \frac{Z_i \check{Y}_i}{\hat{e}(X_i)} - \hat{w}_0(X_i) \frac{(1-Z_i)\check{Y}_i}{1-\hat{e}(X_i)} \right\},
\end{aligned}
$$